CHAPTER 4

Circuit Theorems

The growth in areas of application of electrical circuits has led to an evolution from simple to complex circuits. To handle such complexity, engineers over the years have developed theorems to simplify circuit analysis. These theorems (Thevenin's and Norton's theorems) are applicable to *linear* circuits which are composed of resistors, voltage and current sources.

DEFINITION 4.0.6. **System**:

4.1. Linearity Property

DEFINITION 4.1.1. A **linear system** is a system whose output is linearly related (or directly proportional) to its input[1]. In particular, when we says that the input and output are linearly related, we mean they need to satisfies two properties:

(a) Homogeneous (Scaling): If the input is multiplied by a constant k, then we should observed that the output is also multiplied by k.

(b) Additive: If the inputs are summed then the output are summed.

EXAMPLE 4.1.2. Is the function $f(x) = x^2 + 1$ linear?

[1]The input and output are sometimes referred to as cause and effect, respectively.

EXAMPLE 4.1.3. Is the function $f(x) = 3x + 1$ linear?

4.1.4. A **one-dimensional linear function** is a function of the form

$$y = ax$$

for some constant a.

- For a system, we may call it a **single-input single-output (SISO)** system.
- In radio it is the use of only one antenna both in the transmitter and receiver.

4.1.5. A **multi-dimensional linear function** is a function of the form

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

for some matrix $\mathbf{A}$.

- For a system, when both m and n are greater than one, we may call it a **multiple-input multiple-output system (MIMO)** system.

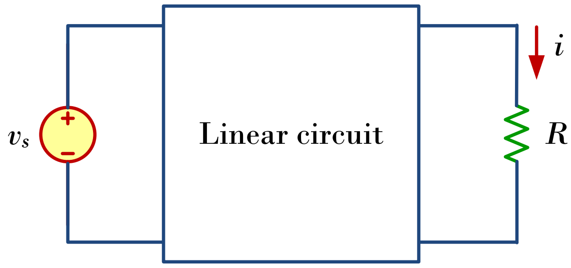
- When $m = n = 1$, we are back to the one-dimensional case in 4.1.4.

EXAMPLE 4.1.6. A resistor is a **linear element** when we consider the current i as its input and the voltage v as its output.

4.1.7. For us, we are considering **linear circuit**.

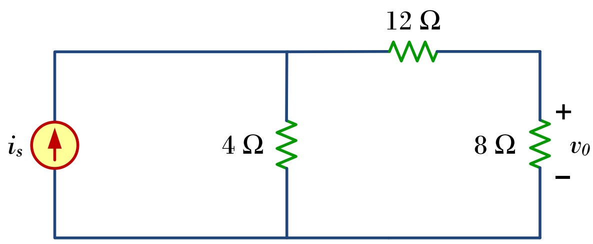
(a) The x_j's will be **all** of the (independent) voltage and current **sources** in the circuit.
(b) Each y_j will correspond to the current or voltage value under investigation.

Example 4.1.8. The circuit below is excited by a voltage source v_s, which serves as the input. Assume that the circuit is linear.



The circuit is terminated by a **load** R. We take the current i through R as the output. Suppose $v_s = 10$ V gives $i = 2$ A. By the assumed linearity, $v_s = 1V$ will give $i = 0.2$ A. By the same token, $i = 1$ mA must be due to $v_s = 5$ mV.

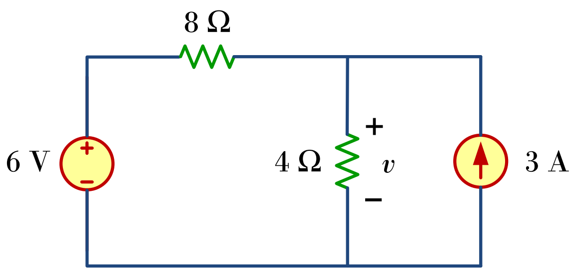
Example 4.1.9. For the circuit below, find v_o when
(a) $i_s = 15$ and (b) $i_s = 30$.



4.1.10. Because $p = i^2R = v^2/R$ (making it a quadratic function rather than a linear one), the relationship between power and voltage (or current) is nonlinear. Therefore, the theorems covered in this chapter are not applicable to power.

4.2. Superposition

EXAMPLE 4.2.1. Find the voltage v in the following circuit.



From the expression of v, observe that there are two contributions.
(a) When I_s acts alone (set $V_s = 0$),

(b) When V_s acts alone (set $I_s = 0$),

Key Idea: Find these contributions from the individual sources and then add them up to get the final answer.

DEFINITION 4.2.2. **Superposition** technique is a way to determine currents and voltages in a circuit that has multiple independent sources by considering the contribution of one source at a time and then add them up.

4.2.3. The **superposition principle** states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

- So, if the circuit has n sources, then we have n cases: "source 1 acting alone", "source 2 acting alone", . . . , "source n acting alone".

4.2.4. To apply the superposition principle, we must keep two things in mind.

(a) We consider one independent source at a time while all other independent source are ***turned off***.[2]
  - Replace other independent voltage sources by 0 V (or short circuits)
  - Replace other independent current sources by 0 A (or open circuits)

  This way we obtain a simpler and more manageable circuit.

(b) Dependent sources are left intact because they are controlled by circuit variable.

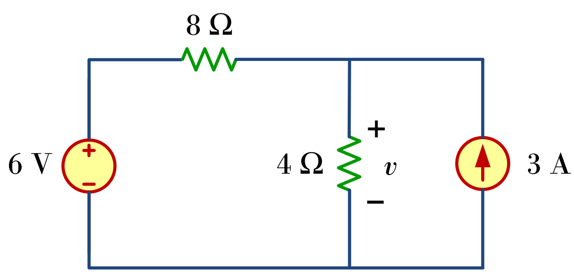
4.2.5. **Steps to Apply Superposition Principles:**

S1: Turn off all independent sources except one source.
Find the output due to that active source. (Here, you may use any technique of your choice.)

S2: Repeat S1 for each of the other independent sources.

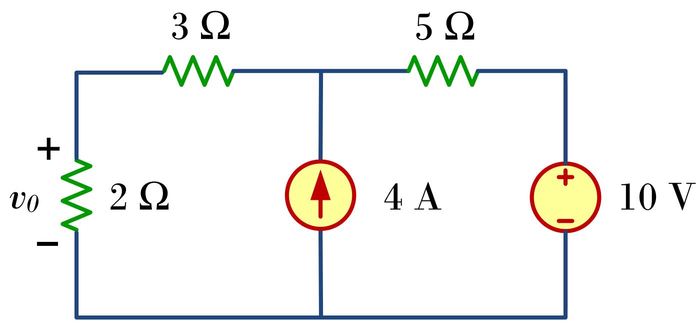
S3: Find the total contribution by adding algebraically all the contributions due to the independent sources.

Example 4.2.6. Back to Example 4.2.1.



[2]Other terms such as killed, made inactive, deadened, or set equal to zero are often used to convey the same idea.

EXAMPLE 4.2.7. Using superposition theorem, find v_o in the following circuit.



4.2.8. Remark on linearity: Keep in mind that superposition is based on linearity. Hence, we cannot find the total power from the power due to each source, because the power absorbed by a resistor depends on the square of the voltage or current and hence it is not linear (e.g. because $5^2 \neq 1^2 + 4^2$).

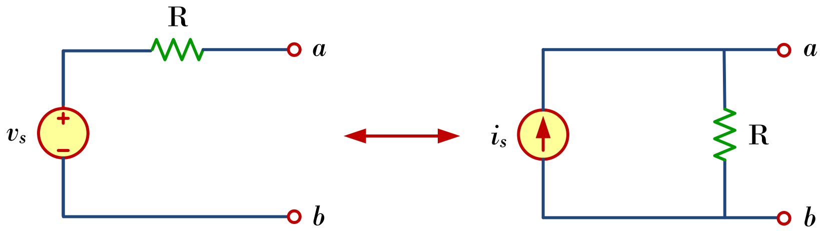
4.2.9. Remark on complexity: Superposition helps reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and of current sources by open circuits.

However, it may very likely involve more work. For example, if the circuit has three independent sources, we may have to analyze three circuits. The advantage is that each of the three circuits is considerably easier to analyze than the original one.

4.3. Source Transformation

We have noticed that series-parallel resistance combination helps simplify circuits. The simplification is done by replacing one part of a circuit by its equivalence.[3] Source transformation is another tool for simplifying circuits.

4.3.1. A **source transformation** is the process of replacing a voltage source in series with a resistor R by a current source in parallel with a resistor R or vice versa.



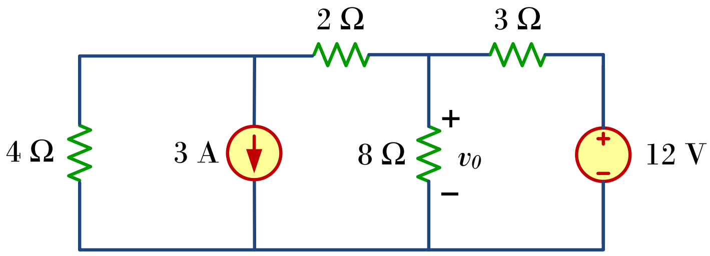
Notice that when terminals $a - b$ are short-circuited, the short-circuit current flowing from a to b is $i_{sc} = v_s/R$ in the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the righthand side. Thus, $v_s/R = i_s$ in order for the two circuits to be equivalent. Hence, source transformation requires that

$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}. \tag{4.2}$$

[3]Recall that an equivalent circuit is one whose $v - i$ characteristics are identical with the original circuit.

4.3.2. Source transformation is usually used repeatedly in combination with the "source combination" and "resistor combination" techniques studied in earlier chapter. In our class, when we say "use source transformation", we actually mean to use all the three techniques above repeatedly to simplify the circuit. At the end, the unknown current or voltage value can usually be obtained by the current divider formula or the voltage divider formula, respectively.

EXAMPLE 4.3.3. Use source transformation to find v_0 in the following circuit:



4.3.4. Cautions:

(a) The "R" in series with the voltage source and the "R" in parallel with the current source are not the same "resistor" even though they have the same value. In particular, the voltage values across them are generally different and the current values through them are generally different.

(b) Keep a circuit variable as a fixed point in the circuit. Do not blindly "transform" and "combine".

(c)

(d) From (4.2), an ideal voltage source with $R = 0$ cannot be replaced by a finite current source. Similarly, an ideal current source with $R = \infty$ cannot be replaced by a finite voltage source.

4.4. Thevenin's Theorem

4.4.1. It often occurs in practice that a particular element in a circuit or a particular part of a circuit is variable (usually called the **load**) while other elements are fixed.

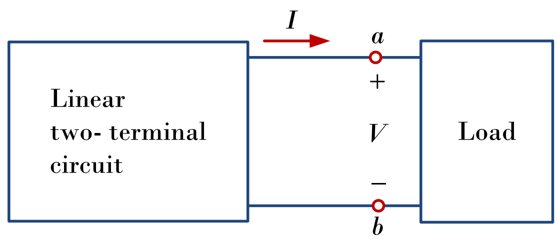
- As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load.

Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenins theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

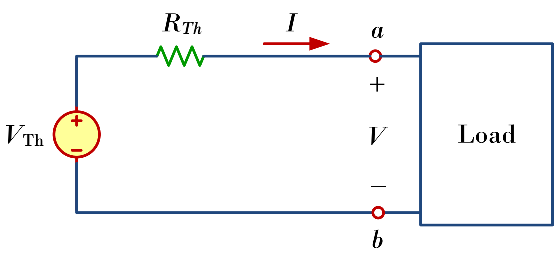
4.4.2. Thevenin's Theorem is an important method to simplify a complicated circuit to a very simple circuit. It states that a circuit can be replaced by an equivalent circuit consisting of an independent voltage source V_{Th} in series with a resistor R_{Th}, where

V_{Th}: the open circuit voltage at the terminal.

R_{Th}: the equivalent resistance at the terminals when the independent sources are turned off.



(a)



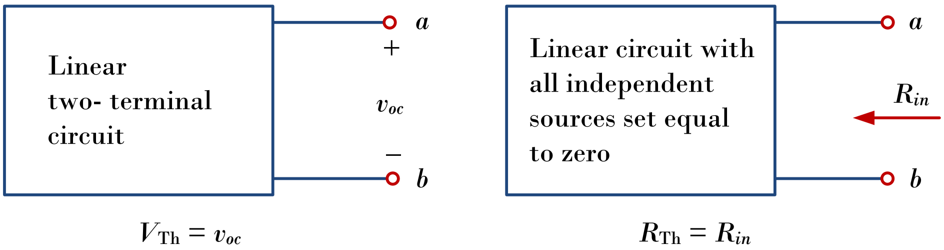
(b)

This theorem allows us to convert a complicated network into a simple circuit consisting of a single voltage source and a single resistor connected in series. The circuit is equivalent in the sense that it looks the same from the outside, that is, it behaves the same electrically as seen by an outside observer connected to terminals a and b.

4.4.3. Steps to Apply Thevenin's theorem.

S1: Find R_{Th}: Turn off all independent sources. R_{Th} is the equivalent resistance of the network looking between terminals a and b.

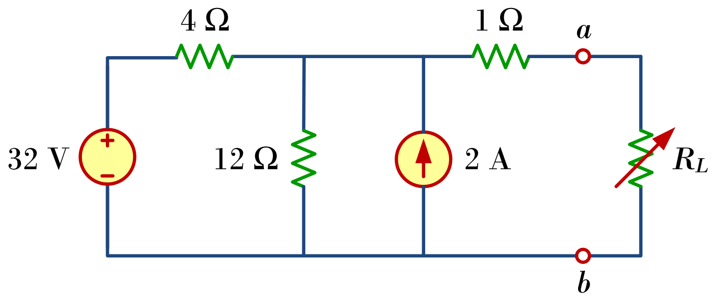
S2: Find V_{Th}: Open the two terminals (remove the load) which you want to find the Thevenin equivalent circuit. V_{Th} is the **open-circuit voltage** across the terminals.



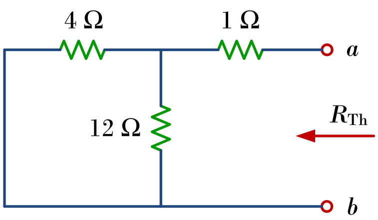
S3: Connect V_{Th} and R_{Th} in series to produce the Thevenin equivalent circuit for the original circuit.

Example 4.4.4. Find the Thevenin equivalent circuits of each of the circuits shown below, to the left of the terminals a-b.

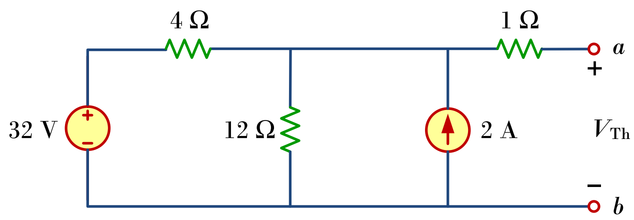
Example 4.4.5. Find the Thevenin equivalent circuit of the circuit shown below, to the left of the terminals a-b. Then find the current through $R_L = 6$, 16, and 36 Ω.



Solution:

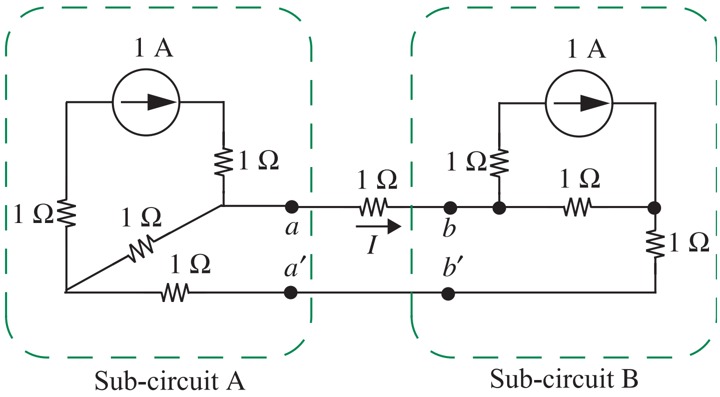


(a)

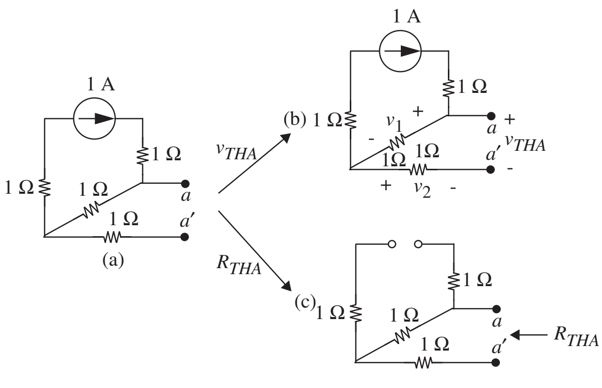


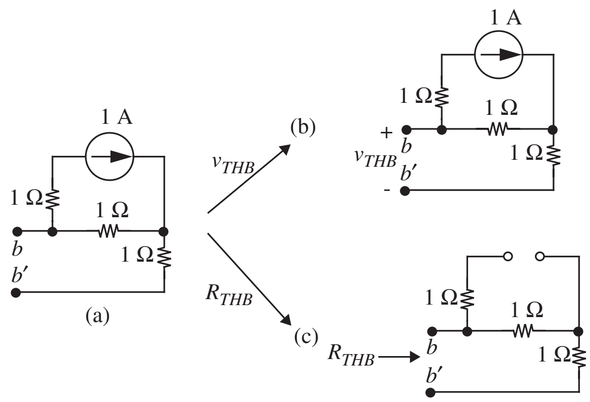
(b)

EXAMPLE 4.4.6. Determine the current I in the branch a-b in the circuit below.



There are many approaches that we can take to obtain the current I. For example, we could apply nodal analysis and determine the node voltages at nodes a and b and thereby determine the current I by Ohm's law. However, here, we will find the Thvenin equivalent circuit for the subcircuit to the left of the aa' terminal pair (Subcircuit A) and for the subcircuit to the right of the bb' terminal pair (Subcircuit B), and then using these equivalent subcircuits to find the current I.



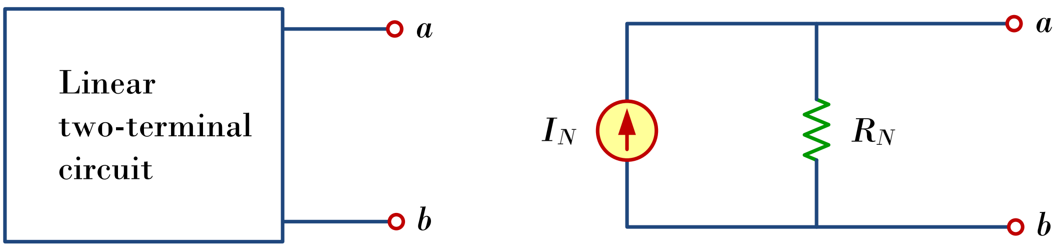


4.5. Norton's Theorem

Norton's Theorem gives an alternative equivalent circuit to Thevenin's Theorem.

4.5.1. **Norton's Theorem:** A circuit can be replaced by an equivalent circuit consisting of a **current source** I_N in **parallel** with a **resistor** R_N, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

Note: $R_N = R_{TH}$ and $I_N = \frac{V_{TH}}{R_{TH}}$. These relations are easily seen via source transformation.[4]

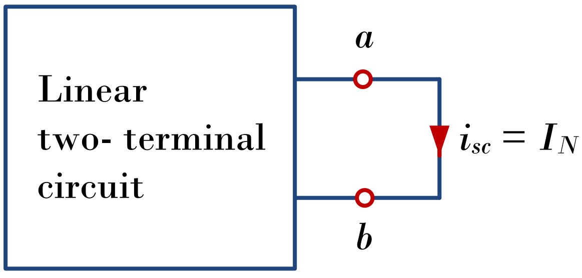


(a) (b)

4.5.2. **Steps to Apply Norton's Theorem**

S1: Find R_N (in the same way we find R_{TH}).

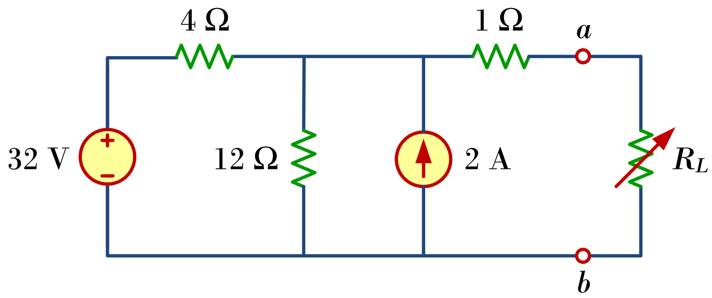
S2: Find I_N: Short circuit terminals a to b. I_N is the current passing through a and b.



S3: Connect I_N and R_N in parallel.

[4]For this reason, source transformation is often called Thevenin-Norton transformation.

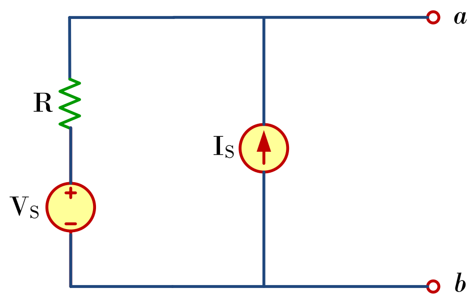
EXAMPLE 4.5.3. Back to the circuit in Example 4.4.5. Directly find the Norton equivalent circuit of the circuit shown below, to the left of the terminals a-b.



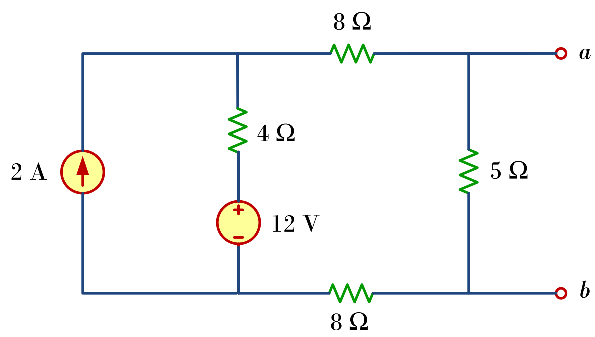
Remark: In our class, to "*directly* find the Norton equivalent circuit" means to follow the steps given in 4.5.2. Similarly, to "*directly* find the Thevenin equivalent circuit" means to follow the steps given in 4.4.3.

Suppose there is no requirement to use the direct technique, then it is quite easy to find the Norton equivalent circuit from the derived Thevenin equivalent circuit in Example 4.4.5.

EXAMPLE 4.5.4. *Directly* find the Norton equivalent circuit of the following circuit at terminals a-b.



EXAMPLE 4.5.5. *Directly* find the Norton equivalent circuit of the circuit in the following figure at terminals a-b.



4.6. Maximum Power Transfer

In many practical situations, a circuit is designed to provide power to a load. In areas such as communications, it is desirable to maximize the power delivered to a load. We now address the problem of delivering the maximum power to a load when given a system with known internal losses.

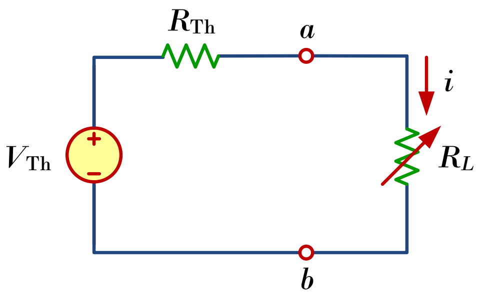
4.6.1. Questions:

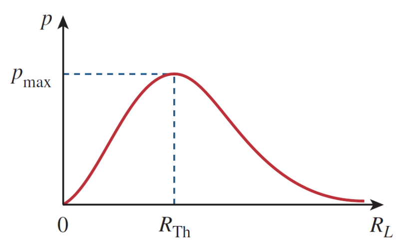
(a) How much power can be transferred to the load under the most ideal conditions?

(b) What is the value of the load resistance that will absorb the maximum power from the source?

4.6.2. If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown below, the power delivered to the load resistor R_L is

$$p = i^2 R_L \quad \text{where} \quad i = \frac{V_{th}}{R_{th} + R_L}$$





The derivative of p with respect to R_L is given by

$$\begin{aligned}\frac{dp}{dR_L} &= 2i\frac{di}{dR_L}R_L + i^2 = 2\frac{V_{th}}{R_{th}+R_L}\left(-\frac{V_{th}}{(R_{th}+R_L)^2}\right) + \left(\frac{V_{th}}{R_{th}+R_L}\right)^2 \\ &= \left(\frac{V_{th}}{R_{th}+R_L}\right)^2\left(-\frac{2R_L}{R_{th}+R_L}+1\right).\end{aligned}$$

We then set this derivative equal to zero and get

$$R_L = R_{TH}.$$

4.6.3. Maximum power transfer occurs when the load resistance R_L equals the Thevenin resistance R_{Th}. The corresponding maximum power transferred to the load R_L equals to

$$p_{\max} = \left(\frac{V_{th}}{R_{th} + R_{th}}\right)^2 R_{th} = \frac{V_{th}^2}{4R_{th}}.$$

EXAMPLE 4.6.4. Connect a load resistor R_L across the circuit in Example 4.4.4. Assume that $R_1 = R_2 = 14\Omega$, $V_s = 56$V, and $I_s = 2$A. Find the value of R_L for maximum power transfer and the corresponding maximum power.

EXAMPLE 4.6.5. Find the value of R_L for maximum power transfer in the circuit below. Find the corresponding maximum power.